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A construction of Hopf algebra cocycles for the Yangian double $DY(\mathfrak{sl}_2)$

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Abstract. We construct a Hopf algebra cocycle for the Yangian double $DY(\mathfrak{sl}_2)$, conjugating Drinfeld's co-product to the usual one. To do this, we factorize the twist between two 'opposite' versions of Drinfeld's co-product, introduced in an earlier work, using the decomposition of the algebra in its negative and non-negative modes sub-algebras.

1. Introduction

The Yangian algebra is a Hopf algebra that was introduced by Drinfeld in [3]. Its quantum double was introduced (without central extension) by Smirnov in [13] in connection with integrable field theory. The doubly extended version of this algebra appeared in [10], and its vertex representations were studied in [9].

The purpose of this paper is to show that Drinfeld's co-product of the Yangian double $DY(\mathfrak{sl}_2)$ [4] is conjugated to the usual one. For this, we construct a Hopf algebra cocycle in the Yangian double $DY(\mathfrak{sl}_2)$. The construction of this cocycle was carried out by Khoroshkin and Tolstoy [11], who used their theory of Cartan–Weyl bases. In their work, the cocycle was expressed as some infinite product. Conversely, our construction relies on elementary arguments, and we express the cocycle as the normal ordering of some exponential. We do not know of a direct way of identifying both expressions.

We note that $DY(\mathfrak{sl}_2)$ is endowed with two variants of Drinfeld's co-product. These co-products are associated with two decompositions of the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2 \otimes \mathbb{C}((z^{-1}))$, the first one being $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, with $\mathfrak{g}_+ = (\mathfrak{h} \otimes \mathbb{C}[z]) \oplus (\mathfrak{n}_+ \otimes \mathbb{C}((z^{-1})))$, $\mathfrak{g}_- = (\mathfrak{h} \otimes z^{-1}\mathbb{C}[[z^{-1}]]) \oplus (\mathfrak{n}_- \otimes \mathbb{C}((z^{-1})))$, and the second one being its transform by the non-trivial Weyl group element of $\mathfrak{sl}_2$. Here $\mathfrak{h}$ and $\mathfrak{n}_\pm$ are the standard Cartan and opposite nilpotent sub-algebras of $\mathfrak{sl}_2$. In [7], we considered Hopf algebras $U_\hbar\mathfrak{g}$ quantizing more general Lie bi-algebra structures associated with curves in higher genus, and showed that they were conjugated by a twist F. The next step was the construction of a deformation $U_\hbar\mathfrak{g}_R$ of the enveloping algebra of an algebra of regular functions with values in $\mathfrak{sl}_2$; in our 'rational' situation, this Lie algebra corresponds to $\mathfrak{sl}_2 \otimes \mathbb{C}[z]$ and $U_\hbar\mathfrak{g}_R$ to the Yangian $Y(\mathfrak{sl}_2)$. This sub-algebra also had the property that

$$\Delta(U_\hbar\mathfrak{g}_R) \subset U_\hbar\mathfrak{g} \otimes U_\hbar\mathfrak{g}_R \qquad \bar{\Delta}(U_\hbar\mathfrak{g}_R) \subset U_\hbar\mathfrak{g}_R \otimes U_\hbar\mathfrak{g}.$$

The last step of [7] was to decompose F as a product $F_2 F_1$, with

$$F_1 \in U_\hbar\mathfrak{g} \otimes U_\hbar\mathfrak{g}_R \qquad F_2 \in U_\hbar\mathfrak{g}_R \otimes U_\hbar\mathfrak{g}$$

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and then to construct a quasi-Hopf algebra structure on $U_{\hbar}\mathfrak{g}_R$ by twisting the co-product Δ by F_1. F_1 and F_2 are constructed by applying to one factor of F a projection of $U_{\hbar}\mathfrak{g}$ on $U_{\hbar}\mathfrak{g}_R$, which is a right $U_{\hbar}\mathfrak{g}_R$-module map. In this construction, the choice of the projection is not unique. Changing the projection has the effect of changing (F_1, F_2) into (uF_1, F_2u^{-1}), for some $u \in U_{\hbar}\mathfrak{g}_R^{\otimes 2}$; this changes the co-product $\mathrm{Ad}(F_1) \circ \Delta$ on $U_{\hbar}\mathfrak{g}_R$ by some twist.

The question naturally arises whether the same technique can be applied in Hopf algebra situations. In this paper, we treat the case of the rational Manin triple $\mathfrak{g} = \mathfrak{g}^{\geqslant 0} \oplus \mathfrak{g}^{<0}$, where $\mathfrak{g}^{\geqslant 0} = \mathfrak{sl}_2 \otimes \mathbb{C}[z]$ and $\mathfrak{g}^{<0} = \mathfrak{sl}_2 \otimes z^{-1}\mathbb{C}[[z^{-1}]]$. In this situation, both $\mathfrak{g}^{\geqslant 0}$ and $\mathfrak{g}^{<0}$ are Lie sub-bi-algebras of $\mathfrak{g}$, and there are also deformations of their enveloping algebras in $DY(\mathfrak{sl}_2)$, $A^{\geqslant 0} = Y(\mathfrak{sl}_2)$ and $A^{<0}$. Therefore, we require that the projection $\Pi_{<0,r}$ be at the same time a right $A^{<0}$-module map. Then it is uniquely determined. We show that the first part, F_1, of the decomposition of F constructed in this way satisfies the Hopf algebra cocycle condition. This is the main result of our paper. The proof of this relies on the following results.

We first prove that the second part, F_2, of the decomposition of F is obtained by applying to F a projection $\Pi_{\geqslant 0,l}$ similar to $\Pi_{<0,r}$ ((28) and (29) later). We give two proofs (sections 3.3, 3.4) of this result, both of them relying on some study of the duality theory within $DY(\mathfrak{sl}_2)$ (section 3.2); the first proof directly applies results from [7]. This enables us to show that the defect of the cocycle identity for F_1 belongs to two spaces with intersection $1 \otimes DY(\mathfrak{sl}_2) \otimes 1$. The fact that the pentagon identity is automatically satisfied by such defects [5] then shows that it is indeed equal to 1.

After we twist by F_1 the universal R-matrix of $DY(\mathfrak{sl}_2)$ associated to Drinfeld's co-product, we obtain a new solution of the Yang–Baxter equation. Applying it to two-dimensional representations of $DY(\mathfrak{sl}_2)$, we construct L-operators satisfying the Yangian exchange (or RLL) relations of [8, 12] (section 5). This connection between Yangian RLL relations and quantum current relations had earlier been obtained in [11] (see [2] in the trigonometric case). After this connection is clarified we are in a position to show (section 6) that F_1 conjugates Δ to the Yangian co-product on $DY(\mathfrak{sl}_2)$.

We will consider an elliptic version of the construction in a separate article [6]. There, we will construct 'twisted cocycles' providing solutions to the dynamical Yang–Baxter equation; this will lead us to the construction of quantum currents of elliptic quantum groups.

The first step towards generalizing our results to the case of a general Lie algebra is to generalize the twist F. This has been done by Reshetikhin (remark 2, later). In the general case, F is then a product of factors corresponding to each simple root; one might expect that these factors satisfy braid relations. We would therefore obtain a 'quantum currents' version of braid group representations. The next step of that generalization would be the study of the duality theory within general Yangian doubles.

2. The Yangian double $DY(\mathfrak{sl}_2)$ and its co-products

The Yangian double is a Hopf algebra that was introduced in [10] (see also [13] in the non-centrally extended case), and is associated to any semisimple Lie algebra $\mathfrak{g}$. In the case where $\mathfrak{g} = \mathfrak{sl}_2$, this algebra is denoted by $DY(\mathfrak{sl}_2)$. We will also denote it by A. It is an algebra over the ring of formal power series in the variable $\overline{h}$, with generators $x_n, n \in \mathbb{Z}$ ($x = e, f, h$), D and K, and the following relations

$$k^+(z)e(w)k^+(z)^{-1} = \frac{z - w + \overline{h}}{z - w}e(w) \tag{1}$$

$$k^+(z)f(w)k^+(z)^{-1} = \frac{z-w}{z-w+\bar{h}}f(w) \tag{2}$$

$$k^-(z)e(w)k^-(z)^{-1} = \frac{z-w+\bar{h}K}{z-w+\bar{h}K+\bar{h}}e(w) \tag{3}$$

$$k^-(z)f(w)k^-(z)^{-1} = \frac{z-w+\bar{h}}{z-w}f(w) \tag{4}$$

$$(z-w-\bar{h})e(z)e(w) = (z-w+\bar{h})e(w)e(z) \tag{5}$$

$$(z-w+\bar{h})f(z)f(w) = (z-w-\bar{h})f(w)f(z) \tag{6}$$

$$[e(z), f(w)] = \frac{1}{\bar{h}}(\delta(z,w)K^+(z) - \delta(z, w-\bar{h}K)K^-(w)^{-1}) \tag{7}$$

$$[K, \text{anything}] = 0 \qquad [k^\pm(z), k^\pm(w)] = 0 \qquad [D, x(z)] = \frac{\mathrm{d}x}{\mathrm{d}z}(z) \qquad x = e, f, k^\pm \tag{8}$$

$$\begin{aligned}(z-w-\bar{h})&(z-w+\bar{h}-\bar{h}K)K^+(z)K^-(w)\\ &= (z-w+\bar{h})(z-w-\bar{h}+\bar{h}K)K^-(w)K^+(z)\end{aligned} \tag{9}$$

where for $x = e, f, h$, we set

$$x^{\geqslant 0}(z) = \sum_{n\geqslant 0} x_n z^{-n-1} \qquad x^{<0}(z) = \sum_{n<0} x_n z^{-n-1} \qquad x(z) = x^{\geqslant 0}(z) + x^{<0}(z)$$

we also set

$$k^+(z) = \exp\left(h_0 \ln\left(\frac{z+\bar{h}}{z}\right) + \sum_{n>0} h_n \frac{z^{-n} - (z+\bar{h})^{-n}}{n}\right) \tag{10}$$

$$K^-(z) = \exp\left(\bar{h}\sum_{n<0} h_n z^{-n-1}\right) \tag{11}$$

and $K^+(z) = k^+(z)k^+(z-\bar{h})$, $K^-(z) = k^-(z)k^-(z-\bar{h})$. In (10) and (11), the arguments of the exponentials are viewed as formal power series in $\bar{h}$, with coefficients in $A \otimes z^{-1}\mathbb{C}((z^{-1}))$ in the first case, and in $A \otimes \mathbb{C}[z]$ in the second one. Finally, $\delta(z,w) = \sum_{n\in\mathbb{Z}} z^n w^{-n-1}$.

Remark 1. The x_n correspond, in the notation of [7], to $x[z^n]$ for $x = e, f, h$ and $n \in \mathbb{Z}$.

The Yangian double Hopf structure Δ_{Yg} is defined as follows (see [10]). Set

$$L^{\geqslant 0}(z) = \begin{pmatrix} 1 & \bar{h}f^{\geqslant 0}(z) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k^+(z-\bar{h}) & 0 \\ 0 & k^+(z)^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \bar{h}e^{\geqslant 0}(z) & 1 \end{pmatrix}$$

and

$$L^{<0}(z) = \begin{pmatrix} 1 & 0 \\ \bar{h}e^{<0}(z-\bar{h}K) & 1 \end{pmatrix} \begin{pmatrix} k^-(z-\bar{h}) & 0 \\ 0 & k^-(z)^{-1} \end{pmatrix} \begin{pmatrix} 1 & \bar{h}f^{<0}(z) \\ 0 & 1 \end{pmatrix}$$

then $L^{\geqslant 0, <0}(z)$ are formal series in z with values in $A \otimes \mathrm{End}(\mathbb{C}^2)$, and we set

$$\Delta_{Yg}(K) = K\otimes 1 + 1\otimes K \qquad \Delta_{Yg}(D) = D\otimes 1 + 1 \otimes D$$

$$(\Delta_{Yg}\otimes 1)L^{\geqslant 0}(z) = L^{\geqslant 0}(z)^{(13)}L^{\geqslant 0}(z)^{(23)} \tag{12}$$

$$(\Delta_{Yg}\otimes 1)L^{<0}(z) = L^{<0}(z-\bar{h}K_1)^{(23)}L^{<0}(z)^{(13)} \tag{13}$$

with $K_1 = K \otimes 1$.

The algebra A can also be endowed with Drinfeld's Hopf structures (Δ, ε, S) and $(\bar{\Delta}, \varepsilon, \bar{S})$. They are given, on the one hand, by the co-product Δ defined by

$$\Delta(k^+(z)) = k^+(z)\otimes k^+(z) \qquad \Delta(K^-(z)) = K^-(z)\otimes K^-(z+\bar{h}K_1) \tag{14}$$

$$\Delta(e(z)) = e(z)\otimes K^+(z) + 1\otimes e(z) \tag{15}$$

$$\Delta(f(z)) = f(z) \otimes 1 + K^-(z)^{-1} \otimes f(z + \hbar K_1) \tag{16}$$

$$\Delta(D) = D \otimes 1 + 1 \otimes D \qquad \Delta(K) = K \otimes 1 + 1 \otimes K \tag{17}$$

the co-unit ε, and the antipode S defined by them; and on the other hand, by the co-product $\bar{\Delta}$ defined by

$$\bar{\Delta}(k^+(z)) = k^+(z) \otimes k^+(z) \qquad \bar{\Delta}(K^-(z)) = K^-(z) \otimes K^-(z + \hbar K_1) \tag{18}$$

$$\bar{\Delta}(e(z)) = e(z - \hbar K_2) \otimes K^-(z - \hbar K_2)^{-1} + 1 \otimes e(z) \tag{19}$$

$$\bar{\Delta}(f(z)) = f(z) \otimes 1 + K^+(z) \otimes f(z) \tag{20}$$

$$\bar{\Delta}(D) = D \otimes 1 + 1 \otimes D \qquad \bar{\Delta}(K) = K \otimes 1 + 1 \otimes K \tag{21}$$

the co-unit ε, and the antipode $\bar{S}$ defined by them.

As we remarked in [7], Δ and $\bar{\Delta}$ are linked by a twist operation. Let us set

$$F = \exp\left(\hbar \sum_{n \in \mathbb{Z}} e_n \otimes f_{-n-1}\right)$$

then we have

$$\bar{\Delta} = \mathrm{Ad}(F) \circ \Delta. \tag{22}$$

(Here and later, we use the notation $\mathrm{Ad}(u)(x) = uxu^{-1}$, for x and u elements of some algebra, with u invertible.)

F satisfies the cocycle condition (see [7])

$$(F \otimes 1)(\Delta \otimes 1)(F) = (1 \otimes F)(1 \otimes \Delta)(F).$$

Remark 2. N Reshetikhin informed us that he obtained the conjugation equation (22) in the general case (that is, with $\mathfrak{sl}_2$ replaced by a semisimple Lie algebra $\mathfrak{g}$). Then F is equal to the product $\prod_{k=1}^{\nu} F_{i_k}$, where $w_0 = s_{i_1} \ldots s_{i_\nu}$ is a decomposition of the longest Weyl group element as a product of simple reflections, and $F_i = q^{\sum_{n \in \mathbb{Z}} e_{i;n} \otimes f_{i;-n-1}}$, with $(e_{i;n})_{n \in \mathbb{Z}}, (f_{i;n})_{n \in \mathbb{Z}}$ the components of the fields corresponding to the ith simple root. The crossed vertex relations seem to imply that all elements F_i commute together. However, this is not quite true: the relations

$$(z - w + \hbar a_{ij})(z - w - \hbar a_{ij}) \left[e_i(z) \otimes f_i(z), e_j(w) \otimes f_j(w)\right] = 0$$

(where a_{ij} are the coefficients of the Cartan matrix) do not imply that the fields $e_i(z) \otimes f_i(z)$ and $e_j(w) \otimes f_j(w)$ commute, but rather the existence of fields $A_{ij}^{\pm}(z)$ such that

$$\left[e_i(z) \otimes f_i(z), e_j(w) \otimes f_j(w)\right] = \delta(z, w + \hbar a_{ij}) A_{ij}^+(z) + \delta(z, w - \hbar a_{ij}) A_{ij}^-(z).$$

It would be interesting to check, using these fields, whether the F_i's satisfy the braid relations. In the same spirit, one is led to construct fields corresponding to non-simple roots using the relation $(z - w - \hbar a_{ij})[e_i(z), (K_j^+ e_j)(w)] = 0$.

3. Decomposition of F

3.1. Sub-algebras of A

We will call $A^{\geqslant 0}$ and $A^{<0}$ the sub-algebras of A generated by D and the $x_n, n \geqslant 0$, respectively by K and the $x_n, n < 0$ (with $x = e, f, h$.) The multiplication induces isomorphisms from $A^{\geqslant 0} \otimes A^{<0}$ and $A^{<0} \otimes A^{\geqslant 0}$ to A; moreover, the intersection of $A^{\geqslant 0}$ with $A^{<0}$ is reduced to $\mathbb{C}1$.

Let $U_\hbar\mathfrak{n}_+$ and $U_\hbar\mathfrak{n}_-$ be the sub-algebras of A generated by the $e_n, n\in\mathbb{Z}$, respectively the $f_n, n\in\mathbb{Z}$.

Let $U_\hbar\mathfrak{n}_\epsilon^{\geqslant 0}$ and $U_\hbar\mathfrak{n}_\epsilon^{<0}$ be the sub-algebras generated by the $x_n, n\geqslant 0$, respectively by the $x_n, n<0$, with $x=e$ for $\epsilon=+$ and $x=f$ for $\epsilon=-$.

The linear maps $U_\hbar\mathfrak{n}_\epsilon^{\geqslant 0}\otimes U_\hbar\mathfrak{n}_\epsilon^{<0}\to U_\hbar\mathfrak{n}_\epsilon$ and $U_\hbar\mathfrak{n}_\epsilon^{<0}\otimes U_\hbar\mathfrak{n}_\epsilon^{\geqslant 0}\to U_\hbar\mathfrak{n}_\epsilon$, defined by the composition of the inclusion with the multiplication, are linear isomorphisms; moreover, the inclusions of algebras $U_\hbar\mathfrak{n}_\epsilon^{\geqslant 0}\subset U_\hbar\mathfrak{n}_\epsilon$ and $U_\hbar\mathfrak{n}_\epsilon^{<0}\subset U_\hbar\mathfrak{n}_\epsilon$ are flat deformations of the inclusions of commutative algebras $\mathbb{C}[x_n, n\geqslant 0]\subset\mathbb{C}[x_n, n\in\mathbb{Z}]$ and $\mathbb{C}[x_n, n<0]\subset\mathbb{C}[x_n, n\in\mathbb{Z}]$ (see [7]).

Remark 3. Relations for generating currents $x^{\geqslant 0}(z)=\sum_{n\geqslant 0}x_n z^{-n-1}$ and $x^{<0}(z)=\sum_{n<0}x_n z^{-n-1}$ of $U_\hbar\mathfrak{n}_\epsilon^{\geqslant 0}$ and $U_\hbar\mathfrak{n}_\epsilon^{<0}$ are

$$(z-w-\hbar)e^\eta(z)e^\eta(w)-(z-w+\hbar)e^\eta(w)e^\eta(z)=-\hbar(e^\eta(z)^2+e^\eta(w)^2)$$

and

$$(z-w+\hbar)f^\eta(z)f^\eta(w)-(z-w-\hbar)f^\eta(w)f^\eta(z)=\hbar(f^\eta(z)^2+f^\eta(w)^2)$$

$\eta\in\{\geqslant 0, <0\}$.

On the other hand, the relations between these currents are

$$(z-w-\hbar)e^\eta(z)e^{\eta'}(w)-(z-w+\hbar)e^{\eta'}(w)e^\eta(z)-\hbar[e^\eta(z)^2+e^{\eta'}(w)^2]=0$$
$$(z-w+\hbar)f^\eta(z)f^{\eta'}(w)-(z-w-\hbar)f^{\eta'}(w)f^\eta(z)+\hbar[f^\eta(z)^2+f^{\eta'}(w)^2]=0$$

if $\{\eta,\eta'\}=\{\geqslant 0, <0\}$.

3.2. Hopf algebra pairings

Let $U_\hbar\mathfrak{h}_+$ be the sub-algebra of A generated by D and the $h_n, n\geqslant 0$, and $U_\hbar\mathfrak{h}_-$ be the sub-algebra of A generated by K and the $h_n, n<0$.

Let $U_\hbar\mathfrak{g}_\pm$ be the sub-algebras of A generated by $U_\hbar\mathfrak{h}_\pm$ and $U_\hbar\mathfrak{n}_\pm$, and $U_\hbar\bar{\mathfrak{g}}_\pm$ the sub-algebras of A generated by $U_\hbar\mathfrak{h}_\mp$ and $U_\hbar\mathfrak{n}_\pm$.

$(U_\hbar\mathfrak{g}_\pm,\Delta)$ are Hopf sub-algebras of (A,Δ); $(U_\hbar\mathfrak{g}_+,\Delta)$ and $(U_\hbar\mathfrak{g}_-,\Delta')$ are dual to each other, and the duality $\langle,\rangle$ is expressed by the rules

$$\langle e_n, f_m\rangle=\frac{1}{\hbar}\delta_{n+m+1,0}\qquad\langle h_a,h_b\rangle=\frac{2}{\hbar}\delta_{a+b+1,0}\qquad\langle D,K\rangle=\frac{1}{\hbar}$$

$n,m\in\mathbb{Z}$, $a\geqslant 0$, $b<0$, the other pairings between generators being trivial.

In a similar way, $(U_\hbar\bar{\mathfrak{g}}_\pm,\bar{\Delta})$ are Hopf sub-algebras of (A,Δ); $(U_\hbar\bar{\mathfrak{g}}_+,\bar{\Delta}')$ and $(U_\hbar\bar{\mathfrak{g}}_-,\bar{\Delta})$ are dual to each other, and the duality $\langle,\rangle'$ is expressed by the rules

$$\langle e_n, f_m\rangle'=\frac{1}{\hbar}\delta_{n+m+1,0}\qquad\langle h_a,h_b\rangle'=\frac{2}{\hbar}\delta_{a+b+1,0}\qquad\langle K,D\rangle'=\frac{1}{\hbar}$$

$n,m\in\mathbb{Z}$, $a\geqslant 0$, $b<0$, the other pairings between generators being trivial.

The restrictions of $\langle,\rangle$ and $\langle,\rangle'$ to $U_\hbar\mathfrak{n}_+\times U_\hbar\mathfrak{n}_-$ coincide and are denoted by $\langle,\rangle_{U_\hbar\mathfrak{n}_\pm}$. Moreover, we have the following lemma.

Lemma 3.1. (See [7].) (1) The annihilator of $U_\hbar\mathfrak{n}_-^{\geqslant 0}$ for $\langle,\rangle_{U_\hbar\mathfrak{n}_\pm}$ is $\sum_{n\geqslant 0}e_n\cdot U_\hbar\mathfrak{n}_+$.

(2) The annihilator of $U_\hbar\mathfrak{n}_+^{<0}$ for $\langle,\rangle_{U_\hbar\mathfrak{n}_\pm}$ is $\sum_{n<0}f_n\cdot U_\hbar\mathfrak{n}_-$.

(3) The annihilator of $U_\hbar\mathfrak{n}_+^{\geqslant 0}$ is $\sum_{n\geqslant 0}U_\hbar\mathfrak{n}_-\cdot f_n$.

(4) The annihilator of $U_\hbar\mathfrak{n}_-^{<0}$ for $\langle,\rangle_{U_\hbar\mathfrak{n}_\pm}$ is $\sum_{n<0}U_\hbar\mathfrak{n}_+\cdot e_n$.

Proof. (1) and (3) are consequences of [7, propositions 6.2], and (2) and (4) are shown in a similar way. □

Finally, the link between F and the pairing $\langle , \rangle_{U_\hbar \mathfrak{n}_\pm}$ can be described as follows. Let us first introduce the notation

$$\langle a, \mathrm{id} \otimes b\rangle_{V,W} = \sum_i a_i \langle a_i', b\rangle_{V,W} \qquad \langle a, b \otimes \mathrm{id}\rangle_{V,W} = \sum_i \langle a_i, b\rangle_{V,W} a_i'$$

for $a \in V^{\otimes 2}$ and $b \in W$, for V, W some vector spaces and $\langle , \rangle_{V,W}$ some pairing between them, a being decomposed as $\sum_i a_i \otimes a_i'$.

Lemma 3.2. (See [7, (66) and (68)].) (1) For any $x \in U_\hbar \mathfrak{n}_+$, we have

$$\langle F, \mathrm{id} \otimes x\rangle_{U_\hbar \mathfrak{n}_\pm} = x.$$

(2) For any $y \in U_\hbar \mathfrak{n}_-$, we have

$$\langle F, y \otimes \mathrm{id}\rangle_{U_\hbar \mathfrak{n}_\pm} = y.$$

3.3. Decomposition of F

Proposition 3.1. There exists a decomposition $F = F_2 F_1$, with $F_1 \in U_\hbar \mathfrak{n}_+^{<0} \otimes U_\hbar \mathfrak{n}_-^{\geqslant 0}$ and $F_2 \in U_\hbar \mathfrak{n}_+^{\geqslant 0} \otimes U_\hbar \mathfrak{n}_-^{<0}$. It is unique up to changes of (F_1, F_2) into $(\lambda F_1, \lambda^{-1} F_2)$, with $\lambda \in \mathbb{C}^\times$.

Proof. Let us denote by $\Pi_{\geqslant 0,l}$, $\Pi_{\geqslant 0,r}$ and by $\Pi_{<0,l}$, $\Pi_{<0,r}$ the linear maps from $U_\hbar \mathfrak{n}_\epsilon$ to $U_\hbar \mathfrak{n}_\epsilon^{\geqslant 0}$ and $U_\hbar \mathfrak{n}_\epsilon^{<0}$ defined by

$$\Pi_{\eta,l}(a_\eta a_{\eta'}) = a_\eta \varepsilon(a_{\eta'}) \qquad \Pi_{\eta,r}(a_{\eta'} a_\eta) = \varepsilon(a_{\eta'}) a_\eta$$

for $\{\eta, \eta'\} = \{\geqslant 0, <0\}$ and $a_\eta \in U_\hbar \mathfrak{n}_\epsilon^\eta$.

Lemma 3.3. (1) $(\Pi_{<0,r} \otimes 1)(F)$ belongs to $U_\hbar \mathfrak{n}_+^{<0} \otimes U_\hbar \mathfrak{n}_-^{\geqslant 0}$.
(2) $(1 \otimes \Pi_{\geqslant 0,r})(F)$ belongs to $U_\hbar \mathfrak{n}_+^{<0} \otimes U_\hbar \mathfrak{n}_-^{\geqslant 0}$.

Proof. (1) $(\Pi_{<0,r} \otimes 1)(F)$ clearly belongs to $U_\hbar \mathfrak{n}_+^{<0} \otimes U_\hbar \mathfrak{n}_-$. On the other hand, we have for any $a \in U\mathfrak{n}_+$ and $n \geqslant 0$

$$\langle (\Pi_{<0,r} \otimes 1)(F), \mathrm{id} \otimes e_n a\rangle = \Pi_{<0,r}(e_n a) = 0.$$

The first equality follows from lemma 3.2(1), and the second from the fact that $\Pi_{<0,r}$ is a left $U_\hbar \mathfrak{n}_+^{\geqslant 0}$-module map. From lemma 3.1(1) it now follows that $(\Pi_{<0,r} \otimes 1)(F)$ also belongs to $U_\hbar \mathfrak{n}_+ \otimes U_\hbar \mathfrak{n}_-^{\geqslant 0}$.

(2) is proved in the same way, using lemma 3.2(2), and lemma 3.1(2). □

Lemma 3.4. $(\Pi_{<0,r} \otimes 1)(F)$ is equal to $(1 \otimes \Pi_{\geqslant 0,r})(F)$.

Proof. Let a_+ belong to $U_\hbar \mathfrak{n}_-^{\geqslant 0}$ and let a_- belong to $U_\hbar \mathfrak{n}_+^{<0}$. Let us compute

$$\langle (\Pi_{<0,r} \otimes 1)(F) - (1 \otimes \Pi_{\geqslant 0,r})(F), a_+ \otimes a_-\rangle_{U_\hbar \mathfrak{n}_\pm^{\otimes 2}}. \tag{23}$$

Due to lemma 3.2, this is equal to $\langle \Pi_{<0,r}(a_-), a_+\rangle_{U_\hbar \mathfrak{n}_\pm} - \langle a_-, \Pi_{\geqslant 0,r}(a_+)\rangle_{U_\hbar \mathfrak{n}_\pm}$. Since $\Pi_{<0,r}(a_-) = a_-$, $\Pi_{\geqslant 0,r}(a_+) = a_+$, (23) is equal to zero.

The pairing $\langle , \rangle_{U_\hbar \mathfrak{n}_\pm}$ is a flat deformation of the symmetric power of the pairing between $\mathbb{C}[z]$ and $z^{-1}\mathbb{C}[[z^{-1}]]$, defined by $\langle f, g\rangle = \mathrm{res}_\infty(fg\,\mathrm{d}z)$. Therefore, it defines an injection of $U_\hbar \mathfrak{n}_-^{\geqslant 0}$ in the dual of $U_\hbar \mathfrak{n}_+^{<0}$ and of $U_\hbar \mathfrak{n}_+^{<0}$ in the dual of $U_\hbar \mathfrak{n}_-^{\geqslant 0}$. That (23) is equal to zero then implies that $(\Pi_{<0,r} \otimes 1)(F) = (1 \otimes \Pi_{\geqslant 0,r})(F)$. □

As $\Pi_{\geqslant 0,r}$ is a right $U_\hbar\mathfrak{n}_+^{\geqslant 0}$-module map, we can apply [7, (74), second statement], with $U_\hbar\mathfrak{g}_R = A^{\geqslant 0}$, and obtain

$$(1 \otimes \Pi_{\geqslant 0,r})(F)F^{-1} \in U_\hbar\mathfrak{n}_+^{\geqslant 0} \otimes U_\hbar\mathfrak{n}_-. \tag{24}$$

We can now apply the arguments of [7, proposition 7.2] to the Hopf algebra (A, Δ'). The role of $U_\hbar\mathfrak{g}_R$ is now played by $A^{<0}$; F is replaced by $F^{(21)}$. The analogue of the second statement of [7, (74)] is then

$$((1 \otimes \Pi_{<0,r})(F^{(21)}))(F^{(21)})^{-1} \in U_\hbar\mathfrak{n}_-^{<0} \otimes U_\hbar\mathfrak{n}_+. \tag{25}$$

We can show in a similar way that

$$(\Pi_{\geqslant 0,l} \otimes 1)(F) = (1 \otimes \Pi_{<0,l})(F)$$

so that this quantity belongs to $U_\hbar\mathfrak{n}_+^{\geqslant 0} \otimes U_\hbar^{<0}\mathfrak{n}_-$, and that

$$F^{-1}(\Pi_{\geqslant 0,l} \otimes 1)(F) \in U_\hbar\mathfrak{n}_+^{<0} \otimes U_\hbar\mathfrak{n}_-^{\geqslant 0}. \tag{26}$$

Consider now the product

$$((\Pi_{\geqslant 0,l} \otimes 1)(F))^{-1}F((\Pi_{<0,r} \otimes 1)(F))^{-1}. \tag{27}$$

Since $(\Pi_{<0,r}\otimes 1)(F) \in U_\hbar\mathfrak{n}_+^{<0}\otimes U_\hbar\mathfrak{n}_-^{\geqslant 0}$, and by (26), this product belongs to $U_\hbar\mathfrak{n}_+^{<0}\otimes U_\hbar\mathfrak{n}_-^{\geqslant 0}$. On the other hand, since $(\Pi_{\geqslant 0,l} \otimes 1)(F) \in U_\hbar\mathfrak{n}_+^{\geqslant 0} \otimes U_\hbar\mathfrak{n}_-^{<0}$, and by (24), it belongs to $U_\hbar\mathfrak{n}_+^{<0} \otimes U_\hbar\mathfrak{n}_-^{\geqslant 0}$. It follows that this product is scalar. Since the constant term in its expansion is equal to one, (27) is equal to one.

Therefore, we can set

$$F_1 = (\Pi_{<0,r} \otimes 1)(F) = (1 \otimes \Pi_{\geqslant 0,r})(F) \tag{28}$$

and

$$F_2 = (\Pi_{\geqslant 0,l} \otimes 1)(F) = (1 \otimes \Pi_{<0,l})(F). \tag{29}$$

□

3.4. Another proof of proposition 3.1

Let us define F_1 and F_2 by

$$F_1 = (\Pi_{<0,r} \otimes 1)(F) \qquad F_2 = (\Pi_{\geqslant 0,l} \otimes 1)(F) \tag{30}$$

and show directly that $F = F_2F_1$. For this, we will consider the linear endomorphism ℓ of $U_\hbar\mathfrak{n}_+$ defined by

$$\ell(x) = \langle F_2F_1, \mathrm{id} \otimes x\rangle_{U_\hbar\mathfrak{n}_\pm}. \tag{31}$$

Let us denote by π the linear map from $U_\hbar\mathfrak{g}_+$ to $U_\hbar\mathfrak{n}_+$, defined by $\pi(tx) = \varepsilon(t)x$, for $x \in U_\hbar\mathfrak{n}_+$, $t \in U_\hbar\mathfrak{h}_+$. Let us also denote by π' the linear map from $U_\hbar\bar{\mathfrak{g}}_+$ to $U_\hbar\mathfrak{n}_+$, defined by $\pi'(x't') = x'\varepsilon(t')$, for $x' \in U_\hbar\mathfrak{n}_+$, $t' \in U_\hbar\mathfrak{h}_-$.

Lemma 3.5. (1) For $y \in U_\hbar\mathfrak{g}_+$, we have

$$\langle F, \mathrm{id} \otimes y\rangle = \pi(y). \tag{32}$$

(2) For $z \in U_\hbar\bar{\mathfrak{g}}_+$, we have

$$\langle F, \mathrm{id} \otimes z\rangle' = \pi'(z). \tag{33}$$

Proof. Let us prove (1). Let us first show that for any $y' \in U_\hbar\mathfrak{n}_-$, we have

$$\langle y', y\rangle = \langle y', \pi(y)\rangle. \tag{34}$$

To prove this, consider the case where $y = t_0 y_0$, $y_0 \in U_\hbar\mathfrak{n}_+$, $t_0 \in U_\hbar\mathfrak{h}_+$. Then $\langle y', y\rangle = \langle \Delta'(y'), t_0 \otimes y_0\rangle_{(2)}$; but $\Delta'(y')$ belongs to $U_\hbar\mathfrak{n}_- \otimes U_\hbar\mathfrak{g}_-$, and for $a \in U_\hbar\mathfrak{h}_+$, $b \in U_\hbar\mathfrak{n}_-$, $\langle a, b\rangle = \varepsilon(a)\varepsilon(b)$. It follows that $\langle y', y\rangle = \langle(\varepsilon \otimes 1)\circ \Delta'(y')\varepsilon(t_0), y_0\rangle = \langle y', \pi(y)\rangle$, so that (34) holds. (32) then follows from (34) and lemma 3.2.

(2) is proved in a similar way. □

We then compute $\ell(x)$ as follows, for $x \in U_\hbar\mathfrak{n}_+$. Set $\Delta(x) = \sum_i x_i' \otimes x_i''$, with $x_i' \in U_\hbar\mathfrak{n}_+$, $x_i'' \in U_\hbar\mathfrak{g}_+$. Then

$$\begin{aligned}\ell(x) &= \sum_i \langle F_2, \mathrm{id}\otimes x_i'\rangle\langle F_1, \mathrm{id}\otimes x_i''\rangle = \sum_i \Pi_{\geqslant 0,l}(x_i')\Pi_{<0,r}(\langle F, \mathrm{id}\otimes x_i''\rangle)\\ &= \sum_i \Pi_{\geqslant 0,l}(x_i')(\Pi_{<0,r}\circ\pi)(x_i''). \end{aligned}\tag{35}$$

We deduce from this expression the following property of ℓ.

Lemma 3.6. ℓ is a left $U_\hbar\mathfrak{n}_+^{\geqslant 0}$-module map.

Proof. $\Pi_{<0,r}\circ\pi$ is defined as follows. Recall that the product operation defines a linear isomorphism from the tensor product $U_\hbar\mathfrak{h}_+ \otimes U_\hbar\mathfrak{n}_+^{\geqslant 0} \otimes U_\hbar\mathfrak{n}_+^{<0}$ onto $U_\hbar\mathfrak{g}_+$. $\Pi_{<0,r}\circ\pi$ is then defined by $(\Pi_{<0,r}\circ\pi)(x) = \varepsilon(tx_{\geqslant 0})x_{<0}$, for x decomposed as $tx_{\geqslant 0}x_{<0}$, $t \in U_\hbar\mathfrak{h}_+$, $x_{\geqslant 0} \in U_\hbar\mathfrak{n}_+^{\geqslant 0}$, $x_{<0} \in U_\hbar\mathfrak{n}_+^{<0}$. On the other hand, denote by $U_\hbar\mathfrak{b}_+$ the subspace of $U_\hbar\mathfrak{g}_+$ corresponding to $U_\hbar\mathfrak{h}_+ \otimes U_\hbar\mathfrak{n}_+^{\geqslant 0}\otimes 1$. We can check that this is a sub-algebra of $U_\hbar\mathfrak{g}_+$. It follows that $\Pi_{<0,r}\circ\pi$ satisfies

$$(\Pi_{<0,r}\circ\pi)(bx) = \varepsilon(b)(\Pi_{<0,r}\circ\pi)(x) \tag{36}$$

for $b \in U_\hbar\mathfrak{b}_+$, $x \in U_\hbar\mathfrak{g}_+$.

Finally, (15) implies that for any $n \in \mathbb{Z}$, $\Delta(e_n) = 1\otimes e_n + \sum_{p\geqslant 0} e_{n-p}\otimes K_p^+$ (we set $K^+(z) = \sum_{p\geqslant 0} K_p^+ z^{-p}$), so that for $n \geqslant 0$ this belongs to $U_\hbar\mathfrak{n}_+ \otimes U_\hbar\mathfrak{b}_+$; it follows that $\Delta(U_\hbar\mathfrak{n}_+^{\geqslant 0}) \subset U_\hbar\mathfrak{n}_+ \otimes U_\hbar\mathfrak{b}_+$.

Let us fix b in $U_\hbar\mathfrak{n}_+^{\geqslant 0}$ and x in $U_\hbar\mathfrak{n}_+$. Set $\Delta(x) = \sum_i x_i'\otimes x_i''$, $x_i' \in U_\hbar\mathfrak{n}_+$, $x_i'' \in U_\hbar\mathfrak{g}_+$, and $\Delta(b) = \sum_j b_j'\otimes b_j''$, $b_j' \in U_\hbar\mathfrak{n}_+$, $b_j'' \in U_\hbar\mathfrak{b}_+$. Then

$$\begin{aligned}\ell(bx) &= \sum_{i,j}\Pi_{\geqslant 0,l}(b_j'x_i')(\Pi_{<0,r}\circ\pi)(b_j''x_i'') = \sum_{i,j}\Pi_{\geqslant 0,l}(b_j'x_i')\varepsilon(b_j'')(\Pi_{<0,r}\circ\pi)(x_i'')\\ &= \sum_i \Pi_{\geqslant 0,l}(bx_i')(\Pi_{<0,r}\circ\pi)(x_i'') = b\sum_i \Pi_{\geqslant 0,l}(x_i')(\Pi_{<0,r}\circ\pi)(x_i'')\\ &= b\ell(x).\end{aligned}$$

The second equality follows from (36), the third from the properties of the co-unit, and the fourth from the fact that $\Pi_{\geqslant 0,r}$ is a left $U_\hbar\mathfrak{n}_+^{\geqslant 0}$-module map. □

Set now $\bar\Delta(x) = \sum_i \bar x_i'\otimes\bar x_i''$, with $\bar x_i' \in U_\hbar\mathfrak{n}_+$, $\bar x_i'' \in U_\hbar\bar{\mathfrak{g}}_+$. Then

$$\begin{aligned}\ell(x) &= \sum_i \langle F_2, \mathrm{id}\otimes\bar x_i''\rangle'\langle F_1, \mathrm{id}\otimes \bar x_i'\rangle' = \sum_i \Pi_{\geqslant 0,l}(\langle F, \mathrm{id}\otimes\bar x_i''\rangle')\Pi_{<0,r}(\bar x_i')\\ &= \sum_i (\Pi_{\geqslant 0,l}\circ\pi')(\bar x_i'')\Pi_{<0,r}(\bar x_i').\end{aligned}\tag{37}$$

We now deduce from this expression the following.

Lemma 3.7. ℓ is a right $U_\hbar\mathfrak{n}_+^{<0}$-module map.

Proof. As above, the product operation defines an isomorphism of vector spaces from $U_\hbar\mathfrak{n}_+^{\geqslant 0} \otimes U_\hbar\mathfrak{n}_+^{<0} \otimes U_\hbar\mathfrak{h}_-$ to $U_\hbar\bar{\mathfrak{g}}_+$. The image by this map of $1 \otimes U_\hbar\mathfrak{n}_+^{<0} \otimes U_\hbar\mathfrak{h}_-$ is a sub-algebra of $U_\hbar\bar{\mathfrak{g}}_+$, that we denote by $U_\hbar\bar{\mathfrak{b}}_+$. $\Pi_{\geqslant 0,l} \circ \pi'$ is then defined by $(\Pi_{\geqslant 0,l} \circ \pi')(x) = \sum_\alpha x_{>0;\alpha}\varepsilon(b_\alpha)$, if x is decomposed as $\sum_\alpha x_{>0;\alpha} b_\alpha$, $x_{>0;\alpha} \in U_\hbar\mathfrak{n}_+^{\geqslant 0}$, $b_\alpha \in U_\hbar\bar{\mathfrak{b}}_+$. Therefore, we have

$$(\Pi_{\geqslant 0,l} \circ \pi')(xb) = (\Pi_{\geqslant 0,l} \circ \pi')(x)\varepsilon(b) \tag{38}$$

for $x \in U_\hbar\bar{\mathfrak{g}}_+, b \in U_\hbar\bar{\mathfrak{b}}_+$.

Finally, (19) implies that for $n > 0$, $\bar{\Delta}(e_{-n}) = \sum_{p\geqslant 0} e_{-n+p} \otimes ((K^-)^{-1})_{-p} + 1 \otimes e_{-n}$ (we set $(K^-)^{-1}(z) = \sum_{p\leqslant 0}((K^-)^{-1})_p z^{-p}$), and so belongs to $U_\hbar\mathfrak{n}_+ \otimes U_\hbar\bar{\mathfrak{b}}_+$; it follows that $\bar{\Delta}(U_\hbar\mathfrak{n}_+) \subset U_\hbar\mathfrak{n}_+ \otimes U_\hbar\bar{\mathfrak{b}}_+$.

Fix x in $U_\hbar\mathfrak{n}_+$, b in $U_\hbar\mathfrak{n}_+^{<0}$, with $\bar{\Delta}(x) = \sum_i \bar{x}'_i \otimes x''_i$, $x'_i \in U_\hbar\mathfrak{n}_+$, $x''_i \in U_\hbar\bar{\mathfrak{g}}_+$, $b'_i \in U_\hbar\mathfrak{n}_+^{<0}$, $b''_i \in U_\hbar\bar{\mathfrak{b}}_+$, $\bar{\Delta}(b) = \sum_j \bar{b}'_j \otimes \bar{b}''_j$. Then

$$\begin{aligned}
\ell(xb) &= \sum_{i,j}(\Pi_{\geqslant 0,l} \circ \pi')(\bar{x}''_i\bar{b}''_j)\Pi_{<0,r}(\bar{x}'_i\bar{b}'_j) = \sum_{i,j}(\Pi_{\geqslant 0,l} \circ \pi')(\bar{x}''_i)\varepsilon(\bar{b}''_j)\Pi_{<0,r}(\bar{x}'_i\bar{b}'_j)\\
&= \sum_{i,j}(\Pi_{\geqslant 0,l} \circ \pi')(\bar{x}''_i)\Pi_{<0,r}(\bar{x}'_i b) = \sum_i(\Pi_{\geqslant 0,l} \circ \pi')(\bar{x}''_i)\Pi_{<0,r}(\bar{x}'_i)b\\
&= \ell(x)b.
\end{aligned}$$

The second equality follows from (36), the third one from the properties of ε, and the fourth one from the fact that $\Pi_{<0,r}$ is a right $U_\hbar\mathfrak{n}_+^{<0}$-module map. □

Let us now prove proposition 3.1. We have $\ell(1) = 1$. Since any element of $U_\hbar\mathfrak{n}_+$ can be expressed as a sum of products $\sum_i x_i^{\geqslant 0} x_i^{<0}$, with $x_i^{\geqslant 0} \in U_\hbar\mathfrak{n}_+^{\geqslant 0}$, $x_i^{<0} \in U_\hbar\mathfrak{n}_+^{<0}$, and by lemmas 3.6 and 3.7, ℓ coincides with the identity. □

4. Cocycle properties

Theorem 4.1. F_1 satisfies the cocycle equation

$$(F_1 \otimes 1)(\Delta \otimes 1)(F_1) = (1 \otimes F_1)(1 \otimes \Delta)(F_1).$$

Proof. First note that

$$\begin{aligned}
&\Delta(A^{\geqslant 0}) \subset A \otimes A^{\geqslant 0} \qquad && \Delta(A^{<0}) \subset A^{<0} \otimes A\\
&\bar{\Delta}(A^{\geqslant 0}) \subset A^{\geqslant 0} \otimes A \qquad && \bar{\Delta}(A^{<0}) \subset A \otimes A^{<0}.
\end{aligned}$$

Let us set

$$\Phi = F_1^{(12)}(\Delta \otimes 1)(F_1)(F_1^{(23)}(1 \otimes \Delta)(F_1))^{-1}$$

we have clearly $\Phi \in A^{<0} \otimes A \otimes A^{\geqslant 0}$. Since we also have

$$\Phi = ((\bar{\Delta} \otimes 1)(F_2)F_2^{(12)})^{-1}(1 \otimes \bar{\Delta})(F_2)F_2^{(23)}$$

we also see that $\Phi \in A^{\geqslant 0} \otimes A \otimes A^{<0}$.

Therefore, $\Phi = 1 \otimes a \otimes 1$, for a certain $a \in A$. On the other hand, as Φ is obtained by twisting a quasi-Hopf structure, it should satisfy the compatibility condition (see [5])

$$(\Delta_1 \otimes \mathrm{id} \otimes \mathrm{id})(\Phi)(\mathrm{id} \otimes \mathrm{id} \otimes \Delta_1)(\Phi) = (\Phi \otimes 1)(\mathrm{id} \otimes \Delta_1 \otimes \mathrm{id})(\Phi)(1 \otimes \Phi)$$

where $\Delta_1 = \mathrm{Ad}(F_1) \circ \Delta$. This implies that

$$1 \otimes a \otimes a \otimes 1 = (1 \otimes a \otimes 1 \otimes 1)(1 \otimes \Delta_1(a) \otimes 1)(1 \otimes 1 \otimes a \otimes 1)$$

and so $\Delta_1(a) = 1$; applying the co-unit to one of the factors of the tensor product where this equality takes place, we obtain $a = 1$. $\square$

Remark 4. Another way to show that Φ is scalar is the following. We can use the third expression of Φ in [7, proposition 7.4] to show that Φ belongs to $A \otimes A^{\geqslant 0} \otimes A$. By writing a similar expression for Φ, we get that $\Phi \in A \otimes A^{<0} \otimes A$. Together with the fact that Φ belongs to $1 \otimes A \otimes 1$, this shows that Φ is scalar.

Remark 5. First-order computations lead us to believe that F_1, and F_2 can be expressed polynomially in terms of the $\mathrm{res}_0(e^{<0}(z) \otimes f^{\geqslant 0}(z))^n \, \mathrm{d}z$, and $\mathrm{res}_0(e^{\geqslant 0}(z) \otimes f^{<0}(z))^n \, \mathrm{d}z$, respectively. Product formulae for $F_{1,2}$ can be found in [11].

5. Yangian *RLL* relations

It follows from theorem 4.1 that we can twist the Hopf algebra structure (A, Δ) by F_1, and get another Hopf algebra structure. The twisted co-product is $\Delta_1 = \mathrm{Ad}(F_1) \circ \Delta$.

Let

$$\mathcal{R} = q^{D \otimes K} q^{\frac{1}{2} \sum_{i \geqslant 0} h_i \otimes h_{-i-1}} q^{\sum_{i \in \mathbb{Z}} e_i \otimes f_{-i-1}} \tag{39}$$

this is the universal R-matrix for (A, Δ) (see [7]). The universal R-matrix for the twisted Hopf algebra (A, Δ_1) is then $\mathcal{R}_1 = F_1^{(21)} \mathcal{R} F_1^{-1}$. We then have the Yang–Baxter equation

$$\mathcal{R}_1^{(12)} \mathcal{R}_1^{(13)} \mathcal{R}_1^{(23)} = \mathcal{R}_1^{(23)} \mathcal{R}_1^{(13)} \mathcal{R}_1^{(12)}. \tag{40}$$

Recall now the formulae for two-dimensional representations of A (see [1]). Let ζ be a formal variable, k_ζ the field of formal Laurent power series $\mathbb{C}((\zeta))$, ∂_ζ the derivation of k_ζ defined as $\mathrm{d}/\mathrm{d}\zeta$, and $k_\zeta[\partial_\zeta]$ the associated ring of differential operators.

Lemma 5.1. There is a morphism of algebras π_ζ from A to $\mathrm{End}(\mathbb{C}^2) \otimes k_\zeta[\partial_\zeta][[\overline{h}]]$, defined by

$$\pi_\zeta(K) = 0 \qquad \pi_\zeta(D) = \mathrm{Id}_{\mathbb{C}^2} \otimes \partial_\zeta$$

$$\pi_\zeta(h_n) = \begin{pmatrix} \left(\dfrac{2}{1 + q^{\partial_z}} z^n \right)(\zeta) & 0 \\ 0 & -\left(\dfrac{2}{1 + q^{-\partial_z}} z^n \right)(\zeta) \end{pmatrix} \qquad n \geqslant 0$$

$$\pi_\zeta(h_n) = \begin{pmatrix} \left(\dfrac{1 - q^{-\partial_z}}{h\partial_z} z^n \right)(\zeta) & 0 \\ 0 & -\left(\dfrac{q^{\partial_z} - 1}{h\partial_z} z^n \right)(\zeta) \end{pmatrix} \qquad n < 0$$

$$\pi_\zeta(e_n) = \begin{pmatrix} 0 & \zeta^n \\ 0 & 0 \end{pmatrix} \qquad \pi_\zeta(f_n) = \begin{pmatrix} 0 & 0 \\ \zeta^n & 0 \end{pmatrix} \qquad n \in \mathbb{Z}.$$

Lemma 5.2. We have

$$(1\otimes\pi_\zeta)(\mathcal{R}_1)=L^{\geqslant 0}(\zeta)\qquad (1\otimes\pi_\zeta)(\mathcal{R}_1^{(21)})=q^{K\partial_\zeta}L^{<0}(\zeta).$$

Proof. Let us denote by $U_{\bar{h}}\mathfrak{n}_\pm^{\geqslant i}$ the linear spans in $U_{\bar{h}}\mathfrak{n}_\pm$ of products of more than i factors e_k, respectively f_k. Then the various $\Pi^\pm_{*,*}$ preserve the $U_{\bar{h}}\mathfrak{n}_\pm^{\geqslant i}$. The formulae (28) for F_1 then imply that F_1 belongs to $1+\bar{h}\sum_{i\geqslant 0}e_{-i-1}\otimes f_i+U_{\bar{h}}\mathfrak{n}_+^{\geqslant 2}\otimes U_{\bar{h}}\mathfrak{n}_-^{\geqslant 2}$. The lemma now follows from the decomposition (39), and from the fact that the $U_{\bar{h}}\mathfrak{n}_\pm^{\geqslant 2}$ are contained in the kernel of π_ζ. □

Lemma 5.3. The image of $\mathcal{R}_1$ by $\pi_\zeta\otimes\pi_{\zeta'}$ is

$$(\pi_\zeta\otimes\pi_{\zeta'})(\mathcal{R}_1)=A(\zeta,\zeta')R^{<0}(\zeta-\zeta')$$

where

$$R^{<0}(z)=\frac{1}{z-\bar{h}}(z\mathrm{Id}_{\mathbb{C}^2\otimes\mathbb{C}^2}-\bar{h}P)$$

where P is the permutation operator of the two factors of $(\mathbb{C}^2)^{\otimes 2}$, and $A(\zeta,\zeta')$ is the formal series

$$\exp\left(\sum_{i\geqslant 0}\left(\frac{1}{\partial}\frac{q^\partial-1}{q^\partial+1}\zeta^i\right)\zeta'^{-i-1}\right).$$

Proof. Since the images by π_ζ and $\pi_{\zeta'}$ of $U_{\bar{h}}\mathfrak{n}_\pm^{\geqslant 2}$ are equal to zero, and using again the fact that F_1 belongs to $1+\bar{h}\sum_{i\geqslant 0}e_{-i-1}\otimes f_i+U_{\bar{h}}\mathfrak{n}_+^{\geqslant 2}\otimes U_{\bar{h}}\mathfrak{n}_-^{\geqslant 2}$, we find that this image is the same as that of

$$\left(1+\bar{h}\sum_{i\geqslant 0}f_i\otimes e_{-i-1}\right)q^{\frac{1}{2}\sum_{i\geqslant 0}h_i\otimes h_{-i-1}}\left(1-\bar{h}\sum_{i\geqslant 0}e_{-i-1}\otimes f_i\right).$$

Let us denote by E_{ij} the endomorphism of $\mathbb{C}^2$ such that $E_{ij}v_\alpha=\delta_{\alpha j}v_i$, where (v_1,v_{-1}) is the standard basis of $\mathbb{C}^2$. We find that

$$\begin{aligned}(\pi_\zeta\otimes\pi_{\zeta'})(\mathcal{R}_1)&=A(\zeta,\zeta')\left(1+\frac{\bar{h}}{\zeta'-\zeta}E_{-1,1}\otimes E_{1,-1}\right)\Big(E_{1,1}\otimes E_{1,1}+E_{-1,-1}\otimes E_{-1,-1}\\&\quad+\frac{\zeta'-\zeta}{\zeta'-\zeta+\bar{h}}E_{1,1}\otimes E_{-1,-1}+\frac{\zeta'-\zeta-\bar{h}}{\zeta'-\zeta}E_{-1,-1}\otimes E_{1,1}\Big)\\&\quad\times\left(1-\frac{\bar{h}}{\zeta-\zeta'}E_{1,-1}\otimes E_{-1,1}\right).\end{aligned}$$

The lemma follows. □

Define $R^{\geqslant 0}(z)$ as the inverse of $R^{<0}(z)$. We have

$$R^{\geqslant 0}(z)=\frac{1}{z+\bar{h}}(z\,\mathrm{Id}_{\mathbb{C}^2\otimes\mathbb{C}^2}+\bar{h}P).$$

Let us now apply to (40) $1\otimes\pi_\zeta\otimes\pi_{\zeta'}$, $\pi_\zeta\otimes\pi_{\zeta'}\otimes 1$ and $\pi_\zeta\otimes\pi_{\zeta'}\otimes 1$. We find the following relations between matrices $L^\pm(\zeta)$.

Proposition 5.1. We have

$$R^\eta(\zeta-\zeta')L^{\eta(1)}(\zeta)L^{\eta(2)}(\zeta')=L^{\eta(2)}(\zeta')L^{\eta(1)}(\zeta)R^\eta(\zeta-\zeta') \tag{41}$$

$$\begin{aligned}&L^{<0(1)}(\zeta)R^{<0}(\zeta-\zeta')L^{\geqslant 0(2)}(\zeta')\\&\quad=L^{\geqslant 0(2)}(\zeta')R^{<0}(\zeta-\zeta'-\bar{h}K)L^{<0(1)}(\zeta)\frac{A(\zeta,\zeta'+\bar{h}K)}{A(\zeta,\zeta')}\\&\quad\eta\in\{\geqslant 0,<0\}.\end{aligned}\tag{42}$$

Remark 6. After analytic continuation in the variables ζ, ζ', we see that $A(\zeta, \zeta')$ only depends on $\zeta - \zeta'$. If we set $A(\zeta, \zeta') = A(\zeta - \zeta')$, we then have

$$A(z)A(z + \hbar) = \frac{z}{z + \hbar}$$

so that A is equal to

$$A(z) = \frac{\Gamma((z/2\hbar) + \frac{1}{2})^2}{\Gamma((z/2\hbar) + 1)\Gamma(z/2\hbar)}.$$

6. F_1 and the Yangian co-product

Since $\Delta(A^{\geqslant 0}) \subset A \otimes A^{\geqslant 0}$ and F_1 and F_1^{-1} belong to $A \otimes A^{\geqslant 0}$, $\Delta_1(A^{\geqslant 0}) \subset A \otimes A^{\geqslant 0}$. On the other hand, $\Delta_1 = \mathrm{Ad}(F_2^{-1}) \circ \bar{\Delta}$; since $\bar{\Delta}(A^{\geqslant 0}) \subset A^{\geqslant 0} \otimes A$, and F_2 and F_2^{-1} belong to $A^{\geqslant 0} \otimes A$, $\Delta_1(A^{\geqslant 0}) \subset A^{\geqslant 0} \otimes A$. This shows that $A^{\geqslant 0}$ is a Hopf sub-algebra of (A, Δ_1).

We can show in the same way that $A^{<0}$ is a Hopf sub-algebra of (A, Δ_1).

Therefore, it is natural to expect that Δ_1 coincides with the Yangian co-product Δ_{Yg}. In this section, we show that this is indeed the case.

Since (A, Δ_1, R_1) is a quasi-triangular Hopf algebra, we have

$$(\Delta_1 \otimes 1)(\mathcal{R}_1) = \mathcal{R}_1^{(13)}\mathcal{R}_1^{(23)} \qquad (1 \otimes \Delta_1)(\mathcal{R}_1) = \mathcal{R}_1^{(13)}\mathcal{R}_1^{(12)}. \tag{43}$$

Apply now $\mathrm{id} \otimes \mathrm{id} \otimes \pi_\zeta$ to the first equation of (43) and $\pi_\zeta \otimes \mathrm{id} \otimes \mathrm{id}$ to the second one. We find

$$(\Delta_1 \otimes 1)L^{\geqslant 0}(\zeta)^{(12)} = L^{\geqslant 0}(\zeta)^{(13)}L^{\geqslant 0}(\zeta)^{(23)} \qquad (\Delta_1 \otimes 1)\mathcal{L}^{<0}(\zeta)^{(12)} = \mathcal{L}^{<0}(\zeta)^{(13)}\mathcal{L}^{<0}(\zeta)^{(23)}$$

where $\mathcal{L}^{<0}(\zeta) = q^{K\partial_\zeta}L^{<0}(\zeta)$; the last equation implies, since $\Delta(K) = K \otimes 1 + 1 \otimes K$, that

$$(\Delta_1 \otimes 1)L^{<0}(\zeta)^{(12)} = L^{<0}(\zeta - \hbar K_1)^{(13)}L^{<0}(\zeta)^{(23)}.$$

Since we also have $[D \otimes 1 + 1 \otimes D, F_1] = 0$ (the algebras $A_\pm$ being $\mathrm{ad}(D)$-invariant), and $[K \otimes 1 + 1 \otimes K, F_1] = 0$, and comparing the above formulae with (12) and (13), we conclude.

Proposition 6.1. $\Delta_1 = \mathrm{Ad}(F_1) \circ \Delta$ coincides with Δ_{Yg}.

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